$$D^4 \cos x = \cos x$$

$$D^5 \cos x = -\sin x$$

We see that the successive derivatives occur in a cycle of length 4 and, in particular, $D^n \cos x = \cos x$ whenever n is a multiple of 4. Therefore

$$D^{24} \cos x = \cos x$$

and, differentiating three more times, we have

$$D^{27} \cos x = \sin x \qquad \square$$

We have seen that one application of second and third derivatives occurs in analyzing the motion of objects using acceleration and jerk. We will investigate another application of second derivatives in Exercise 62 and in Section 4.3, where we show how knowledge of f'' gives us information about the shape of the graph of f. In Chapter 11 we will see how second and higher derivatives enable us to represent functions as sums of infinite series.
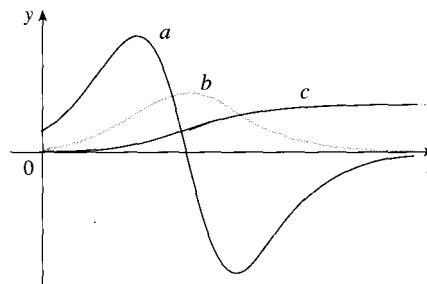
3.7 Exercises

1. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.
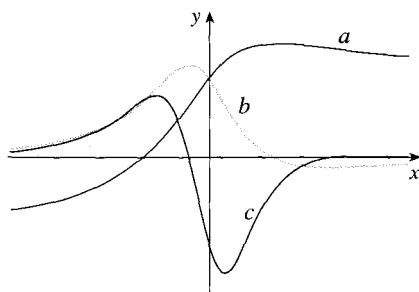

2. The figure shows graphs of f, f', f'', and f'''. Identify each curve, and explain your choices.
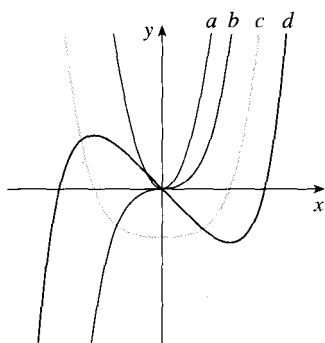
3. The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your choices.

4. The figure shows the graphs of four functions. One is the position function of a car, one is the velocity of the car, one is its acceleration, and one is its jerk. Identify each curve, and explain your choices.
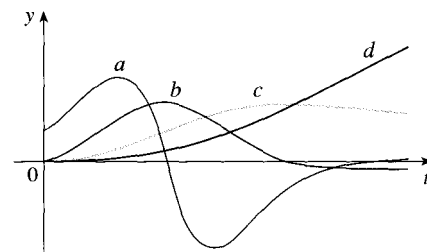
5–20 □ Find the first and second derivatives of the function.

5. $f(x) = x^5 + 6x^2 - 7x$

6. $f(t) = t^8 - 7t^6 + 2t^4$

7. $y = \cos 2\theta$

8. $y = \theta \sin \theta$

9. $h(x) = \sqrt{x^2 + 1}$

10. $G(r) = \sqrt{r} + \sqrt[3]{r}$

11. $F(s) = (3s + 5)^8$

12. $g(u) = \dfrac{1}{\sqrt{1 - u}}$

13. $y = \dfrac{x}{1 - x}$

14. $y = xe^{cx}$

15. $y = (1 - x^2)^{3/4}$

16. $y = \dfrac{x^2}{x + 1}$

17. $H(t) = \tan 3t$

18. $g(s) = s^2 \cos s$

19. $g(t) = t^3 e^{5t}$

20. $h(x) = \tan^{-1}(x^2)$

21. (a) If $f(x) = 2 \cos x + \sin^2 x$, find $f'(x)$ and $f''(x)$.
   (b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

22. (a) If $f(x) = e^x - x^3$, find $f'(x)$ and $f''(x)$.
   (b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

23–24 □ Find y'''.

23. $y = \sqrt{2x + 3}$

24. $y = \dfrac{1 - x}{1 + x}$

25. If $f(x) = (2 - 3x)^{-1/2}$, find $f(0)$, $f'(0)$, $f''(0)$, and $f'''(0)$.

26. If $g(t) = (2 - t^2)^6$, find $g(0)$, $g'(0)$, $g''(0)$, and $g'''(0)$.

27. If $f(\theta) = \cot \theta$, find $f'''(\pi/6)$.

28. If $g(x) = \sec x$, find $g'''(\pi/4)$.

29–32 □ Find y'' by implicit differentiation.

29. $x^3 + y^3 = 1$

30. $\sqrt{x} + \sqrt{y} = 1$

31. $x^2 + xy + y^2 = 1$

32. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

33–37 □ Find a formula for $f^{(n)}(x)$.

33. $f(x) = x^n$

34. $f(x) = \dfrac{1}{(1 - x)^2}$

35. $f(x) = e^{2x}$

36. $f(x) = \sqrt{x}$

37. $f(x) = \dfrac{1}{3x^3}$

38–40 □ Find the given derivative by finding the first few derivatives and observing the pattern that occurs.

38. $D^{99} \sin x$

39. $D^{50} \cos 2x$

40. $D^{1000} xe^{-x}$

41. A car starts from rest and the graph of its position function is shown in the figure, where s is measured in feet and t in seconds. Use it to graph the velocity and estimate the acceleration at $t = 2$ seconds from the velocity graph. Then sketch a graph of the acceleration function.
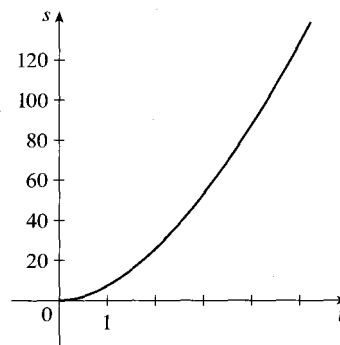
42. (a) The graph of a position function of a car is shown, where s is measured in feet and t in seconds. Use it to graph the velocity and acceleration of the car. What is the acceleration at $t = 10$ seconds?
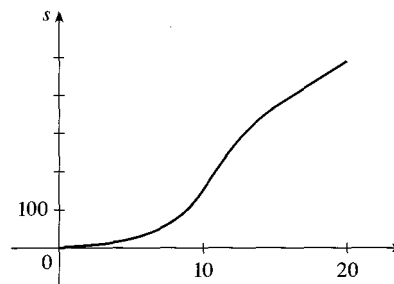
   (b) Use the acceleration curve from part (a) to estimate the jerk at $t = 10$ seconds. What are the units for jerk?

43–46 □ The equation of motion is given for a particle, where s is in meters and t is in seconds. Find (a) the velocity and acceleration as functions of t, (b) the acceleration after 1 second, and (c) the acceleration at the instants when the velocity is 0.

43. $s = t^3 - 3t$

44. $s = t^2 - t + 1$

45. $s = \sin 2\pi t$

46. $s = 2t^3 - 7t^2 + 4t + 1$

47–48 □ An equation of motion is given, where s is in meters and t in seconds. Find (a) the times at which the acceleration is 0 and (b) the displacement and velocity at these times.

47. $s = t^4 - 4t^3 + 2$

48. $s = 2t^3 - 9t^2$

49. A particle moves according to a law of motion
$s = f(t) = t^3 - 12t^2 + 36t$, $t \geq 0$, where t is measured in
seconds and s in meters.
(a) Find the acceleration at time t and after 3 s.
(b) Graph the position, velocity, and acceleration functions
for $0 \leq t \leq 8$.
(c) When is the particle speeding up? When is it slowing
down?

50. A particle moves along the x-axis, its position at time t given
by $x(t) = t/(1 + t^2)$, $t \geq 0$, where t is measured in seconds
and x in meters.
(a) Find the acceleration at time t. When is it 0?
(b) Graph the position, velocity, and acceleration functions for
$0 \leq t \leq 4$.
(c) When is the particle speeding up? When is it slowing
down?

51. A mass attached to a vertical spring has position function given
by $y(t) = A \sin \omega t$, where A is the amplitude of its oscillations
and ω is a constant.
(a) Find the velocity and acceleration as functions of time.
(b) Show that the acceleration is proportional to the displace-
ment y.
(c) Show that the speed is a maximum when the acceleration
is 0.

52. A particle moves along a straight line with displacement $s(t)$,
velocity $v(t)$, and acceleration $a(t)$. Show that

$$a(t) = v(t) \frac{dv}{ds}$$

Explain the difference between the meanings of the derivatives
dv/dt and dv/ds.

53. Find a second-degree polynomial P such that $P(2) = 5$,
$P'(2) = 3$, and $P''(2) = 2$.

54. Find a third-degree polynomial Q such that $Q(1) = 1$,
$Q'(1) = 3$, $Q''(1) = 6$, and $Q'''(1) = 12$.

55. The equation $y'' + y' - 2y = \sin x$ is called a **differential
equation** because it involves an unknown function y and its
derivatives y' and y''. Find constants A and B such that the
function $y = A \sin x + B \cos x$ satisfies this equation. (Differ-
ential equations will be studied in detail in Chapter 9.)

56. Find constants A, B, and C such that the function
$y = Ax^2 + Bx + C$ satisfies the differential equation
$y'' + y' - 2y = x^2$.

57. For what values of r does the function $y = e^{rx}$ satisfy the equa-
tion $y'' + 5y' - 6y = 0$?

58. Find the values of λ for which $y = e^{\lambda x}$ satisfies the equation
$y + y' = y''$.

59–61 □ The function g is a twice differentiable function. Find f''
in terms of g, g', and g''.

59. $f(x) = xg(x^2)$

60. $f(x) = \dfrac{g(x)}{x}$

61. $f(x) = g(\sqrt{x})$

. . . .

62. If $f(x) = 3x^5 - 10x^3 + 5$, graph both f and f''. On what
intervals is $f''(x) > 0$? On those intervals, how is the graph of
f related to its tangent lines? What about the intervals where
$f''(x) < 0$?

63. (a) Compute the first few derivatives of the function
$f(x) = 1/(x^2 + x)$ until you see that the computations are
becoming algebraically unmanageable.
(b) Use the identity

$$\frac{1}{x(x + 1)} = \frac{1}{x} - \frac{1}{x + 1}$$

to compute the derivatives much more easily. Then find an
expression for $f^{(n)}(x)$. This method of splitting up a fraction
in terms of simpler fractions, called *partial fractions*, will
be pursued further in Section 7.4.

64. (a) If $F(x) = f(x)g(x)$, where f and g have derivatives of all
orders, show that

$$F'' = f''g + 2f'g' + fg''$$

(b) Find similar formulas for F''' and $F^{(4)}$.
(c) Guess a formula for $F^{(n)}$.

65. If $y = f(u)$ and $u = g(x)$, where f and g are twice differen-
tiable functions, show that

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

66. If $y = f(u)$ and $u = g(x)$, where f and g possess third deriva-
tives, find a formula for d^3y/dx^3 similar to the one given in
Exercise 65.